1. (10%) Find the following limit or prove it doesn't exist

$$\lim_{x,y\to 0}\frac{1-\cos(x+2y)}{\sin(xy)}.$$

Denote the function under limit by $f(x,y)$. Consider $g_k(x)=f(x,kx)$. If $\lim_{x\to 0} g_k(x)$ depends on k, the limit we are interested in does not exist. **(3 points)**.

$\lim_{x\to 0} g_k(x) = \frac{1-\cos(x+2kx)}{\sin(kx^2)}$. This limit can be evaluated using Taylor expansions or via l'Hopitale rule. Second approach leads to **(5 points)**

$$\lim_{x\to 0} g_k(x) = \frac{\frac{(1+2k)^2x^2}{2}}{kx^2} = \frac{(1+2k)^2}{2k}.$$

As we can see, this limit depends on k, so the limit we are interested in does not exist. **(2 points)**.

2. (10%) For the following sequence

$$x_n = \frac{(-1)^n}{n} + \frac{1+(-1)^n}{2}$$

find

$$\sup x_n, \inf x_n, \limsup_{n\to\infty} x_n, \liminf_{n\to\infty} x_n, \lim_{n\to\infty} x_n.$$

Calculate several first values of the sequence

$$x_1 = -1+0=-1, x_2 = \frac{1}{2}+1=\frac{3}{2}, x_3=-\frac{1}{3}+0=-\frac{1}{3}, x_4=\frac{1}{4}+1=\frac{5}{4}.$$

(2 points) As we can see, the subsequence $x_n=-\frac{1}{n}, n=2k-1, k\in\mathbb{N}$ tends to zero and increases and the subsequence $x_n = 1+\frac{1}{n}, n=2k, k\in\mathbb{N}$ tends to 1 and decreases **(3 points)**.

Hence supremum $\sup x_n = 3/2$ and is achieved for $n=2$ **(1 point)**. The upper limit of x_n equals 1 and corresponds to the subsequence with even indices **(1 point)**.

Infinum $\inf x_n = -1$ and is achieved for $n=1$ **(1 point)**. The lower limit of x_n equals to 0 and corresponds to the subsequence with odd indices **(1 point)**. Since lower and upper limits are different, the limit does not exist **(1 point)**.

3. (10%) Peter wants to congratulate his mother and grandmother on the International Women's Day. He decided to write congratulations on homemade cards. The borders of homemade cards are curves that satisfy the equations: The card for mother

$$\begin{pmatrix} x & y\end{pmatrix}\begin{pmatrix}3 & 1\\ 1 & 3\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = 100$$

The card for grandmother

$$\begin{pmatrix} x & y\end{pmatrix}\begin{pmatrix}3.5 & \sqrt{3}/2\\ \sqrt{3}/2 & 2.5\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = 100$$

All values given in centimeters.

What size envelopes should Peter buy to put these cards in? Suggest minimal possible size for each envelope.

The borders of cards are second order curves. To find the exact form we find the eigenvalues of the matrix. For both cases the canonic form is $4u^2+2v^2=100$.

It follows that major axis is $\sqrt{50}$ and minor axis is 5. Envelope of size $10\times 2\sqrt{50}$ is ok!

4. Find the orthogonal projection of the vector $f = (1,\ 2,\ 3,\ 4,\ 5)$ on the linear subspace $L \subset \mathbb{R}^5$ generated by vectors $g_1 = (1,\ 1,\ 1,\ 1,\ 1)$, $g_2 = (0,\ 0,\ 0,\ 1,\ 1)$ and $g_3 = (0,\ 0,\ 0,\ 0,\ 1)$. Find the normal vector.

Let us represent vector f in the following form $f = g + h$, where $g \in L$ and $h \in L^{\perp}$. This means that $g = \alpha_1 g_1 + \alpha_2 g_2 + \alpha_3 g_3$, where $\alpha_1,\ \alpha_2,\ \alpha_3 \in \mathbb{R}$, and $\forall j = 1, 2, 3\colon\ \langle g_j, f\rangle = 0$. Thus we arrive at the following system

$$\begin{cases} \alpha_1\langle g_1, g_1\rangle + \alpha_2\langle g_1, g_2\rangle + \alpha_3\langle g_1, g_3\rangle = \langle g_1, f\rangle, \\ \alpha_1\langle g_2, g_1\rangle + \alpha_2\langle g_2, g_2\rangle + \alpha_3\langle g_2, g_3\rangle = \langle g_2, f\rangle, \\ \alpha_1\langle g_3, g_1\rangle + \alpha_2\langle g_3, g_2\rangle + \alpha_3\langle g_3, g_3\rangle = \langle g_3, f\rangle, \end{cases}$$

which is equivalent to

$$\begin{cases} 5\alpha_1 + 2\alpha_2 + 1\alpha_3 = 15, \\ 2\alpha_1 + 2\alpha_2 + 1\alpha_3 = 9, \\ 1\alpha_1 + 1\alpha_2 + 1\alpha_3 = 5. \end{cases}$$

Solving this system, we get unique solution $\alpha_1 = 2$, $\alpha_2 = 2$, $\alpha_3 = 1$. So we have

$$g = \alpha_1 g_1 + \alpha_2 g_2 + \alpha_3 g_3 = 2g_1 + 2g_2 + 1g_3 = (2,\ 2,\ 2,\ 4,\ 5),$$

and $h = f - g = (-1,\ 0,\ 1,\ 0,\ 0)$.

Answer: $g = (2,\ 2,\ 2,\ 4,\ 5)$, $h = (-1,\ 0,\ 1,\ 0,\ 0)$.

5. (10%) Find conditional extrema of the function

$$F(x_1, x_2, x_3) = 0.5x_1^2 + 5x_1x_2 + 0.5x_2^2 + x_3^2$$

subject to $x_1^2 + x_2^2 + x_3^2 = 1$.

The objective function can be written in the form

$$F(x_1, x_2, x_3) = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 0.5 & 2.5 & 0 \\ 2.5 & 0.5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x^T A x$$

The constraint has the form $x^T x = 1$. The Lagrangian function is given by

$$L(x, \lambda) = F(x) + \lambda(x^T x - 1)$$

First order conditions are

$$\begin{cases} \frac{\partial L}{\partial x} = 2(Ax - \lambda x) = 0 \\ \frac{\partial L}{\partial \lambda} = x^T x - 1 = 0 \end{cases}$$

The solutions of this system are eigenvectors and eigenvalues of the matrix A.

Let's find them:

$$\begin{cases} F(x_A^T) = \lambda_A = 3,\ x_A^T = \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \end{pmatrix} \\ F(x_B^T) = \lambda_B = -2,\ x_B^T = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \end{pmatrix} \\ F(x_C^T) = \lambda_C = 1,\ x_C^T = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \end{cases}$$

We can restate the problem in the canonic way. Find extrema of the function $3u_1^2 - 2u_2^2 + u_3^2$ subject to $u_1^2 + u_2^2 + u_3^2 = 1$. From this form we see that A — is the point of maximum, B — the point of minimum, and C is not an extremum.

6. (10%) Solve the following system of differential equations:

$$\begin{cases} \ddot{x} = 2x - 3y, \\ \ddot{y} = x - 2y. \end{cases} \tag{1}$$

Answer must be represented in real form.

Let us express x from the second equation of the system and substitute it into the first equation of the system. So we get equation $y^{(4)} + 2y^{(2)} = 2(y^{(2)} + 2y) - 3y$, which is equivalent to

$$y^{(4)} - y = 0. \tag{2}$$

Let us solve the corresponding characteristic equation $\lambda^4 - 1 = 0$. We get the following roots: $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = i$, $\lambda_4 = -i$. Thus the solution of the equation has the following form

$$y(t) = C_1 e^t + C_2 e^{-t} + C_3 \cos t + C_4 \sin t, \text{ where } C_1, \ldots, C_4 \in \mathbb{R}. \tag{3}$$

Substituting y into the second equation of the system, we arrive at

$$x(t) = 3C_1 e^t + 3C_2 e^{-t} + C_3 \cos t + C_4 \sin t, \text{ where } C_1, \ldots, C_4 \in \mathbb{R}.$$

Answer:

$$\begin{cases} x(t) = 3C_1 e^t + 3C_2 e^{-t} + C_3 \cos t + C_4 \sin t \\ y(t) = C_1 e^t + C_2 e^{-t} + C_3 \cos t + C_4 \sin t, \end{cases}$$

where all $C_i \in \mathbb{R}$.

7. Consider some statistical test with 1% probability of Type I error and power of 60%. Researcher assumes that a priori (before the test) probability of the null hypothesis is 0.5.

(a) (8%) If the test indicates a rejection of the null hypothesis, what is the probability that the null is false?

Let $R = \{\text{reject the null}\}$, $A = \{\text{fail to reject the null}\}$, $T = \{\text{the null is true}\}$, $F = \{\text{the null is false}\}$.
We know that $\mathbb{P}(T) = \mathbb{P}(F) = 0.5$, $\mathbb{P}(R|T) = 0.01$, $\mathbb{P}(A|F) = 0.4$. (1 point)
We should find $\mathbb{P}(F|R)$.
Using Bayes rule, we get: (2 points)

$$\mathbb{P}(F|R) = \frac{\mathbb{P}(R|F)\,\mathbb{P}(F)}{\mathbb{P}(R)}$$

Denominator: (3 points)

$$\mathbb{P}(R) = \mathbb{P}(R|T)\,\mathbb{P}(T) + \mathbb{P}(R|F)\,\mathbb{P}(F) = 0.01 \cdot 0.5 + 0.6 \cdot 0.5 = 0.305$$

Thus (2 points)

$$\mathbb{P}(F|R) = \frac{0.6 \cdot 0.5}{0.305} = \frac{0.3}{0.305} \approx 0.984$$

(b) (2%) Briefly explain how in general probabilities of Type I and Type II errors are connected?

For a given test and constant sample size higher probability of Type I error means lower probability of Type II error. Change of sample size or test type may affect both probabilities in the same direction.

8. A magic shield protects Earth from alien invasion. The shield only works when it is powered by a power generator. Once one generator breaks down, another generator is turned on, and there is no way to repair the broken generator. Once the last generator breaks down, aliens start the invasion.

As of today, there are two generators on Earth: one is currently powering the shield, the second is kept in reserve. It is known that working times of generators are independent and have exponential distributions with means 5 and 10 years for first and second generator respectively.

(a) (1%) Find the probability that the first generator will last for at least 2 years more if it is known, that it was turned on 5 years ago.

Let X be the working time of the first generator, Y - of the second. Then, we need to find $\mathbb{P}(X > 7|X > 5)$. For exponential distribution, $\mathbb{P}(X > 7|X > 5) = \mathbb{P}(X > 2)$. As $\mathbb{E}(X) = 5$, $F_X(x) = 1 - e^{-\lambda_X x}, x > 0$, where $\lambda_X = \frac{1}{5}$. So, the probability sought is $1 - F_X(2) = e^{-\frac{2}{5}}$

(b) (5%) Let Z be the time the Earth shield will last (time until the last generator breaks down). Find the probability density function of Z

$Z = X + Y$
We can use convolution:

$$f_Z(z) = \int_0^z f_X(z-t)f_Y(t)dt = \int_0^z \lambda_X\lambda_Y e^{-(\lambda_X(z-t)+\lambda_Y t)}dt = \frac{\lambda_X\lambda_Y}{\lambda_X-\lambda_Y}(e^{-\lambda_Y z} - e^{-\lambda_X z})$$

Then we can substitute $\lambda_X = \frac{1}{5}$ and $\lambda_Y = \frac{1}{10}$ to get the final answer.

(c) (4%) The magic shield of Mars works in another fashion: it requires BOTH generators to work simultaneously. If at least one generator breaks down, the shield disappears. Find the distribution of working time of Mars shield with the same two generators as were used on Earth.

$Z = \min(X, Y)$
So, we can calculate

$$F_Z(z) = \mathbb{P}(Z \le z) = 1 - \mathbb{P}(\min(X,Y) > z) = 1 - \mathbb{P}(X > z)\,\mathbb{P}(Y > z) = 1 - e^{-\lambda_X z}e^{-\lambda_Y z} = 1 - e^{-(\lambda_X+\lambda_Y)z}$$

Hence, Z has exponential distribution with $\lambda_Z = \lambda_X + \lambda_Y$.

9. Anna and Bella found a biased coin, which lands on heads with unknown probability p. On the first day they flipped the coin 100 times. On the second day — 200 times. On the third day — 400 times.

Anna noted the total number of heads for the first and second day: 120 times. Bella noted the total number of heads for the second and the third day: 300 times.

(a) (6%) Please help Anna and Bella to construct an unbiased estimate of p with minimal variance.

Let's denote the number of heads by S_1, S_2 and S_3, and number of throws by n_1, n_2 and n_3. Anna alone would construct the estimator $\hat{p}_a = \frac{S_1+S_2}{n_1+n_2}$. Bella alone would construct the estimator $\hat{p}_b = \frac{S_2+S_3}{n_2+n_3}$. Our goal is to minimize variance of an unbiased estimator:

$$\mathbb{V}\mathrm{ar}(\hat{p}) = \mathbb{V}\mathrm{ar}(\alpha\hat{p}_a + (1-\alpha)\hat{p}_b) \to \min_\alpha$$

Or explicitely

$$Q(\alpha) = p(1-p)\left(\alpha^2\frac{1}{n_1+n_2} + (1-\alpha)^2\frac{1}{n_2+n_3} + 2\alpha(1-\alpha)\frac{n_2}{(n_1+n_2)(n_2+n_3)}\right)$$

Weights α and $1-\alpha$ assure unbiased estimator.
The function is quadratic and non-negative, so the derivative is equal to zero at the minimum. The optimal point is

$$\alpha^* = \frac{n_1}{n_1+n_3}$$

Estimates: $\hat{p}_a = 0.4$, $\hat{p}_b = 0.5$, $\alpha = 0.4$, $\hat{p} = 0.46$.
Grading: 4 points — correct objective function, 2 points — optimal point.

(b) (2%) Estimate the variance of the estimator you have constructed.

Just plug in α in the formula for variance. The minimal variance is equal to

$$\frac{n_1n_2 + n_2n_3 + n_1n_3}{(n_1+n_2)(n_1+n_3)(n_2+n_3)}p(1-p)$$

Estimate of the variance is $\widehat{\mathbb{V}\mathrm{ar}}(\hat{p}) \approx 0.0458$.

(c) (2%) Find the 95% confidence interval for the unknown p.

Usual formula may be applied

$$[\hat{p} - 1.96se(\hat{p}); \hat{p} + 1.96se(\hat{p})]$$

With $\hat{p} = 0.46$ and $se(\hat{p}) \approx 0.214$.

10. Let X_1, X_2, ..., X_{100} be a random sample from normal distribution $\mathcal{N}(0; \sigma^2)$. The parameter σ^2 is unknown. We observe that $\sum_{i=1}^{100} y_i^2 = 200$.

(a) (4%) Write the log-likelihood function and find the maximum likelihood estimate of σ^2.

Log likelihood funciton is

$$\ell(\sigma^2) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}\sum y_i^2$$

The optimal point is

$$\hat{\sigma}^2 = \frac{\sum y_i^2}{n}$$

The estimate is $\hat{\sigma}^2 = 2$.

Grading: log-likelihood function = 2 points, estimate = 2 points. Estimation using formula for unbiased estimator, $\sum y_i^2/(n-1)$, gives 1 point.

(b) (4%) Find the Fisher information and estimate it.

Second derivative is equal to

$$H = \frac{n}{2\sigma^4} - \frac{1}{\sigma^6}\sum y_i^2$$

Fisher information is

$$I = -\mathbb{E}(H) = \frac{n}{2\sigma^4}$$

Estimate of Fisher information is $100/8$.

Grading: second derivative = 1 point, Fisher information = 2 points, estimate of = 1 point.

(c) (2%) Find approximate 95% confidence interval for σ^2.

The estimate of variance of estimator is $\hat{I}^{-1} = 8/100$.

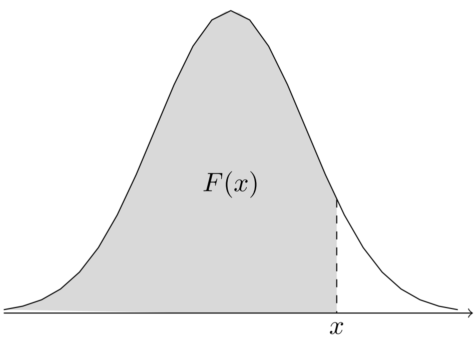
Usual formula may be applied

$$[\hat{\sigma}^2 - 1.96se(\hat{\sigma}^2); \hat{\sigma}^2 + 1.96se(\hat{\sigma}^2)],$$

where $\hat{\sigma}^2 = 2$ and $se(\hat{\sigma}^2) \approx 0.28$.

One may also use chi-squared distribution here.

Рис. 1: Distribution function of a standard normal random variable



x	$F(x)$	x	$F(x)$	x	$F(x)$	x	$F(x)$
0.050	0.520	0.750	0.773	1.450	0.926	2.150	0.984
0.100	0.540	0.800	0.788	1.500	0.933	2.200	0.986
0.150	0.560	0.850	0.802	1.550	0.939	2.250	0.988
0.200	0.579	0.900	0.816	1.600	0.945	2.300	0.989
0.250	0.599	0.950	0.829	1.650	0.951	2.350	0.991
0.300	0.618	1.000	0.841	1.700	0.955	2.400	0.992
0.350	0.637	1.050	0.853	1.750	0.960	2.450	0.993
0.400	0.655	1.100	0.864	1.800	0.964	2.500	0.994
0.450	0.674	1.150	0.875	1.850	0.968	2.550	0.995
0.500	0.691	1.200	0.885	1.900	0.971	2.600	0.995
0.550	0.709	1.250	0.894	1.950	0.974	2.650	0.996
0.600	0.726	1.300	0.903	2.000	0.977	2.700	0.997
0.650	0.742	1.350	0.911	2.050	0.980	2.750	0.997
0.700	0.758	1.400	0.919	2.100	0.982	2.800	0.997

Good luck!